On the infrared behaviour
of the (singlet) Higgs propagator

M. Consoli

Istituto Nazionale di Fisica Nucleare, Sezione di Catania

Corso Italia 57, 95129 Catania, Italy

Abstract

We present a simple semi-perturbative argument in favour of a peculiar infrared behaviour of the (singlet) Higgs propagator. On the basis of 'triviality' one expects a continuum limit with a two-point function $\Gamma_2(q) \to (q^2 + M_h^2)$. However, this is not valid in the limit $q \to 0$ where one actually finds a singular behaviour. This is in agreement with both non-perturbative analyses of the effective potential and with lattice computations of the propagator and of the zero-momentum susceptibility in the broken phase. The singular behaviour persists in an O(N) continuous-symmetry theory, the case first pointed out by Symanzik, and supports the existence of an extremely weak $1/r$ potential that does not disappear when coupling the scalar fields to gauge bosons.

1.   Introduction

The generally accepted 'triviality' [1] of $\lambda\Phi^4$ theories in four space-time dimensions is usually interpreted within leading-order perturbation theory in a very intuitive way. One starts with the perturbative one-loop $\beta-$function

$$\beta_{\text{pert}}(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3) \tag{1.1}$$

and integrates the differential equation

$$\frac{dx}{x} = \frac{d\lambda}{\beta(\lambda)} \tag{1.2}$$

between a fixed energy scale $x = \mu$ and the 'Landau pole' $x = \Lambda$ where $\lambda(x) = +\infty$. In this way, at energy scales $\mathcal{O}(\mu)$, the theory is governed by a 1PI 4-point function

$$\lambda \equiv \lambda(\mu) = \frac{32\pi^2}{3\ln\frac{\Lambda^2}{\mu^2}} \tag{1.3}$$

that would vanish in the continuum limit where the ultraviolet cutoff $\Lambda \to \infty$ within all loop diagrams. In this picture, differently from the original renormalization-group approach where $\lambda(\mu)$ is changed along a given integral curve of Eq.(1.2) by changing μ, one considers all possible integral curves at the same time. After that, one changes λ, at fixed μ, depending on the magnitude of the Landau pole associated with the various integral curves. In this way, the 'bare' theory is always defined in the infinite-coupling limit, i.e. as in an Ising model. However, at any finite scale μ, the continuum limit corresponds to a vanishingly small interaction strength so that perturbation theory in the small parameter $\lambda(\mu)$ should provide very accurate predictions for low-energy physical observables.

By adopting this view of 'triviality' in the spontaneously broken phase of $(\lambda\Phi^4)_4$, the euclidean propagator for the Higgs particle should approach the form

$$\tilde{G}(q) \to \frac{Z_{\text{prop}}}{q^2 + M_h^2} \tag{1.4}$$

with a residue

$$Z_{\text{prop}} = 1 - |\mathcal{O}(\lambda)| \to 1 \tag{1.5}$$

consistent with the Källen-Lehmann decomposition that dictates the spectral function $\rho_h(s)$ to approach $\delta(s - M_h^2)$ in the continuum limit of a 'trivial' theory.

This simple picture neglects, however, that the origin of spontaneous symmetry breaking is not necessarily of perturbative nature. Indeed, one may be faced with non-analytic $1/\lambda$

effects so that, even with an infinitesimal two-body interaction strength, there may be non-perturbative effects. To understand the possible implications, one should remember the case of superconductivity. This is due to the basic instability of a normal Fermi system in the presence of an *infinitesimally small* attractive interaction between electrons. Due to the presence of a very large density of quantum states near the Fermi surface, superconductivity is a non-perturbative phenomenon.

In the case of spontaneous symmetry breaking with an elementary scalar field the delicate issue concerns the limit $q \to 0$ that requires some care in the case of a macroscopic occupation of the same quantum state, i.e. of Bose-Einstein condensation. Just for this reason, and contrary to the most naive expectations, the approach to the continuum limit in the spontaneously broken phase of $(\lambda \Phi^4)_4$ theories may contain unexpected features. These are discovered whenever one takes seriously 'triviality' as a technical statement controlling the approach to the continuum theory and, thus, supporting the idea of a trivially free fluctuation field with a gaussian quantum measure for $\Lambda \to \infty$. In fact, as pointed out in refs.[2, 3, 4, 5], assuming gaussian (and post-gaussian) ansatz for the ground state wave functional, one finds

$$\frac{\Gamma_2(0)}{M_h^2} = \mathcal{O}(1/\ln \Lambda) \to 0 \tag{1.6}$$

once the zero-momentum two-point function is computed through the effective potential

$$\Gamma_2(0) \equiv \left. \frac{d^2 V_{\text{eff}}}{d\phi^2} \right|_{\phi=v} \tag{1.7}$$

As pointed out in refs.[2, 3, 4, 5], Eq. (1.6) requires a non-trivial re-scaling

$$Z \equiv Z_\phi = \mathcal{O}(\ln \Lambda) \to \infty \tag{1.8}$$

of the vacuum field ϕ

$$\phi_R^2 \equiv \frac{\phi^2}{Z_\phi} \tag{1.9}$$

in order to match the quadratic shape of the effective potential with the physical mass M_h defined from the $q \neq 0$ behaviour of the propagator

$$\left. \frac{d^2 V_{\text{eff}}}{d\phi_R^2} \right|_{\phi_R=v_R} \equiv M_h^2. \tag{1.10}$$

As such, $Z = Z_\phi$ is quite distinct from the 'trivial' re-scaling $Z = Z_{\text{prop}}$ in Eq.(1.5), and one may obtain a continuum limit where, although M_h vanishes in units of the bare v, *both* M_h and v_R are finite quantities (with potentially important implications for the commonly quoted upper bounds on the Higgs mass from 'triviality' [6]).

The result in Eq.(1.6) is also striking for the following reason. The euclidean value $q = 0$ corresponds, indeed, to the single point $(q_o, \mathbf{q}) = 0$ in the continuum theory. However, in the cutoff theory, Eq.(1.6) implies that there is a *region* of 3-momenta say $\mathbf{q}^2 \ll \mathcal{O}(1/\ln \Lambda)M_h^2$ where the energy spectrum is *not* $\tilde{E}(\mathbf{q}) = \sqrt{\mathbf{q}^2 + M_h^2}$. This region, although infinitesimal in units of M_h, can have a physical meaning (for $M_h = \mathcal{O}(10^2)$ GeV, think of the values $|\mathbf{q}| \ll 10^{-5}$ eV/c corresponding to wavelengths much larger than 1 cm).

The previous result was obtained in the formalism of the gaussian (and post-gaussian) approximations to the effective potential. In the next section, we shall outline a simple semi-perturbative argument that leads to the same conclusions and can help to understand in a more intuitive way the singular nature of the limit $q \to 0$ when approaching the continuum theory from the broken phase.

2. A simple semi-perturbative calculation

Let us consider a one-component $\lambda \Phi^4$ theory

$$\mathcal{L} = \frac{1}{2}(\partial \Phi)^2 - U(\Phi) \tag{2.1}$$

with a classical potential ($\lambda > 0$)

$$U(\Phi) = \frac{1}{2}m_B^2 \Phi^2 + \frac{\lambda}{4!}\Phi^4 \tag{2.2}$$

Classically, non-vanishing constant field configurations $\phi = \pm v$ occur where $U'(\pm v) = 0$. At these values, one gets a quadratic shape

$$U''(\pm v) = \frac{\lambda v^2}{3} \equiv M_h^2 \tag{2.3}$$

that represents the well known classical result for the Higgs mass.

In the quantum theory, the question of vacuum stability is more subtle and one has to replace $U(\phi)$ with the quantum effective potential $V_{\text{eff}}(\phi)$. However, the basic expectation is that the excitation spectrum of the broken phase will maintain the Lorentz-covariant form $\tilde{E}(\mathbf{q}) = \sqrt{\mathbf{q}^2 + M_h^2}$ down to $\mathbf{q} = 0$ so that M_h should coincide with $\tilde{E}(0)$ that represents the energy-gap of the broken phase.

To check this prediction, let us consider the one-loop structure of the gap-equation for the euclidean two-point function

$$\tilde{G}^{-1}(q) \equiv \Gamma_2(q) = q^2 + m_B^2 + \frac{\lambda \phi^2}{2} + \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k) - \frac{\lambda^2 \phi^2}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k)\tilde{G}(k+q) \tag{2.4}$$

3

together with the vanishing of the one-loop tadpoles, i.e.

$$T(\phi) \equiv m_B^2 + \frac{\lambda\phi^2}{6} + \frac{\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k) = 0 \tag{2.5}$$

Eqs.(2.4) and (2.5) can be understood, for instance, as coupled minimization equations of the effective potential for composite operators introduced by Cornwall, Jackiw and Tomboulis [7]. As such, they are non-perturbative, being equivalent to the *all-order* resummation of one-loop graphs with the tree-level propagator in the external background field ϕ

$$\tilde{G}(q)_{\text{tree}} = \frac{1}{q^2 + m_B^2 + \frac{\lambda\phi^2}{2}} \tag{2.6}$$

Notice that for $\phi \neq 0$, the gap-equation would also contain one-particle reducible terms (i.e. not contained in $\Gamma_2(q)$) proportional to the zero-momentum limit of the shifted field propagator $\tilde{G}(0)$. In this sense, by considering the 1PI gap-equation, we are *assuming* a non-singular zero-momentum limit, even for $\phi = \pm v$ where $\pm v$ are the solutions of (2.5) and represent, to this order, the minima of the effective potential.

The possibility to solve simultaneously Eqs.(2.4) and (2.5), in the continuum limit of the regularized theory by using Eqs. (1.3) - (1.5), amounts to describe spontaneous symmetry breaking as a *quantum* phenomenon of vacuum instability consistently with (the intuitive interpretation of) rigorous quantum field theoretical results on $(\lambda\Phi^4)_4$ theories.

By using (2.5) in (2.4) we get

$$\tilde{G}^{-1}(q) \equiv \Gamma_2(q) = q^2 + \frac{\lambda v^2}{3} A(q) \tag{2.7}$$

where

$$A(q) = 1 - \frac{3\lambda}{2} \int \frac{d^4k}{(2\pi)^4} \tilde{G}(k)\tilde{G}(k+q) \tag{2.8}$$

Notice that a form of the propagator as in Eq.(1.4) can be a solution of Eq.(2.4) only if the coupling λ is understood as an infinitesimally small quantity, i.e. as in Eq.(1.3) with $\mu \sim M_h$. In this case, for $G(q) \sim \frac{1}{q^2}$ at large euclidean q^2, one finds $A(q) = A(0) + \mathcal{O}(1/\ln\Lambda)$ and one indeed gets a form $\Gamma_2(q) = q^2 + const$, up to terms vanishing in the continuum limit where $\lambda \to 0$. In this sense, renormalized perturbation theory should be considered an external input whose overall consistency with the all-order Eqs.(2.4) and (2.5) can only be checked *a posteriori*.

By using Eq. (1.4) in Eqs.(2.7) and (2.8) we obtain the leading-order expression

$$Z_{\text{prop}} = \frac{1}{1 + \frac{\lambda v^2}{3M_h^2} \frac{\lambda Z_{\text{prop}}^2}{64\pi^2}} \tag{2.9}$$

4

and the estimate of the Higgs mass

$$M_h^2 = \frac{\lambda v^2 Z_{\text{prop}}^2}{3}\left(1 - \frac{3\lambda Z_{\text{prop}}^2}{32\pi^2}\ln\frac{\Lambda^2}{M_h^2}\right) \tag{2.10}$$

At this point we find a strong contradiction. Indeed, by using Eq. (1.3) and assuming (1.5) we obtain from (2.10)

$$M_h^2 = \mathcal{O}(\lambda^2 v^2) \tag{2.11}$$

that when inserted in Eq.(2.9), does *not* produce a $Z_{\text{prop}} \to 1$ when $\Lambda \to \infty$.

It is true that in a more conventional fixed-order perturbative calculation one would tend to regard (2.10) as the first two terms in the expansion of the renormalized coupling constant at a scale M_h (i.e. the renormalization-group improved version of the classical result (2.3)). However, this cannot work in our case for two reasons. First, as anticipated, Eqs.(2.4) and (2.5) resum one-loop graphs with the tree propagator (2.6) to all orders, so that our results cannot be considered *fixed order* calculations. Second, using the simple form of the propagator (1.4) to solve Eq.(2.4) requires an infinitesimal λ, as in (1.3). However, if one tries to improve on (2.4) and (2.5), by introducing *genuine* two-loop terms, the intuitive interpretation of 'triviality' based on Eqs.(1.1)-(1.5) is destroyed. This is due to the presence of a (spurious) ultraviolet fixed point at finite coupling in the two-loop perturbative $\beta-$function [8] so that, beyond a leading-order calculation, there are no reasons for the 1PI four point function $\lambda(\mu)$ to vanish in the continuum limit as in Eq.(1.3) and for Eqs. (1.4) and (1.5) to be valid.

The previous results suggest that, contrary to our assumption, either 'triviality' cannot be understood within leading-order perturbation theory, or one is faced with a singular $\tilde{G}(0)$ when $\phi = \pm v$. Namely, a result $M_h^2 = \mathcal{O}(\lambda v^2)$ from Eqs. (2.5) and (2.4), in order Eq.(2.9) to agree with (1.5), can only be obtained if there are one-particle reducible contributions to the propagator. These, at a generic value of ϕ, are proportional to the combination

$$R = T(\phi)\tilde{G}(0) \tag{2.12}$$

and the case $\phi = \pm v$ should be understood as a limiting procedure due to a possible divergence in $\tilde{G}(0)$. To this end, let us introduce $\phi^2 \equiv v^2(1 + \delta)$ with $|\delta| \ll 1$, and define

$$g \equiv \tilde{G}(0)\frac{\lambda v^2}{3} \tag{2.13}$$

Now a non vanishing R requires

$$g \sim \frac{1}{\delta} \tag{2.14}$$

implying that, at $\phi = \pm v$ where $\delta = 0$, there is a mode whose energy vanishes for $\mathbf{q} \to 0$. For this reason, the energy-gap of the broken phase, obtained from $\tilde{E}(\mathbf{q})$ for $\mathbf{q} \to 0$, cannot be M_h.

3. The zero-momentum discontinuity

We believe that the discrepancy we have pointed out, is related to the subtleties of Bose-Einstein condensation [9] in an *almost* ideal gas [10] where there is a macroscopic occupation of the same quantum state. This phenomenon cannot be *fully* understood in a purely perturbative manner. However, as anticipated in the Introduction, the peculiarity of the zero-momentum limit is found in other approaches such as:

   i) variational evaluations of the effective potential [2, 3, 4, 5]

   ii) lattice computations of the propagator and of the zero-momentum susceptibility $\chi^{-1} \equiv \Gamma_2(0)$ in the broken phase [11]. These show that Eq.(1.4), although valid at higher momenta, has *non-perturbative* corrections for $q \to 0$. Indeed, these become larger and larger by approaching the continuum limit and therefore cannot represent perturbative $\mathcal{O}(\lambda)$ effects

   Due to this qualitative agreement with other approaches, the semi-perturbative calculation of sect.2 can be considered a reasonably self-contained treatment of the basic zero-momentum discontinuity.

   The discrepancy persists in the case of an O(N) continuous-symmetry $\lambda\Phi^4$ theory. To this end, one can check with a non-perturbative Gaussian effective-action approach [14]. In this case, the minimization conditions of the gaussian effective potential provide suitable relations that replace the 'triviality pattern' Eqs.(1.3) and (1.5) and where the equivalent of $T(\phi)$ plays the role of a mass term for the Goldstone bosons. By neglecting one-particle reducible contributions in the $\sigma-$field propagator, one finds (N-1) massless fields with propagator $D_\pi(q) = 1/q^2$ and a $\sigma-$field two-point function of the type as in Eqs.(2.7) and (2.8). By following the steps of ref.[14], it is not difficult to check that one gets in this way the same discrepancy as in sect.2 for the mass of the $\sigma-$field. This can also be understood since, for the Goldstone bosons, the non-perturbative wave functional of ref.[14] reproduces 'triviality' and is *exact*. In fact, it yields (N-1) non-interacting fields that decouple from each other and from the $\sigma-$field. As a consequence, one gets effectively the same type of $\sigma - \sigma$ interactions as in our discrete-symmetry case.

   On the other hand, in the case of an O(N) continuous symmetry, the singular nature of the zero-momentum limit of the singlet-Higgs propagator is well known. It was first pointed out by Symanzik for the linear σ-model [15], and later on by Patashinsky and Pokrowsky [16]

6

and Anishetty et al. [17].

Symanzik's analysis for the $\sigma-$ field, although purely perturbative, displays the essential features of the phenomenon, i.e. the perturbative contradiction between finite 1PI vertex diagrams and finite Green's functions that introduces the zero-momentum discontinuity. Just for this reason, he introduced two different notations, namely $\Gamma_\sigma(0) \equiv M^2$ and $\Gamma_\sigma(q^2) \equiv (q^2 + \bar{M}^2)$, to emphasize that the limit $q \to 0$ is not defined.

From ref.[16], on the other hand, one can get a better feeling of what is actually going on. In fact, in the case of a spontaneously broken O(N) symmetry, the longitudinal susceptibility is found [16]

$$\chi_{||}(\mathbf{q}) \sim \frac{1}{|\mathbf{q}|} \ arctg\frac{|\mathbf{q}|}{2\kappa} \tag{3.1}$$

where

$$\kappa^2 \sim (\phi - v)^2 \tag{3.2}$$

Now, for any $\phi \neq v$, the limit $\mathbf{q} \to 0$ yields a finite result. However, just in the case $\phi = v$, $\chi_{||}(\mathbf{q})$ becomes singular when $\mathbf{q} \to 0$ as in our case.

Our results show no qualitative difference with respect to the continuous-symmetry case of ref.[14] and, therefore, the agreement is not surprising. Indeed, the zero-momentum discontinuity does *not* depend on the existence of a continuous symmetry of the Lagrangian. Rather, its physical origin has to be searched in the presence of the scalar condensate, i.e. in the phenomenon of Bose-Einstein condensation that leads to a gap-less mode and to a long-range $1/r$ potential [18, 19].

This can be understood as follows. As discussed in refs.[12, 13, 3, 10], variational approximations to V_{eff} describe spontaneous symmetry breaking as an *infinitesimally* weak first-order transition. This occurs when the mass-gap at $\phi = 0$, say $0 \leq m^2 \leq m_c^2$, is still positive, but in a 'hierarchical' relation

$$\frac{m^2}{M_h^2} = \mathcal{O}(\frac{1}{\ln \frac{\Lambda^2}{\mu^2}}) \tag{3.3}$$

with the mass scale of the broken phase [20].

As discussed in ref.[18], the gap-less mode with $\tilde{E}(\mathbf{q}) = const.|\mathbf{q}|$ is associated with the infinitesimal region of momenta

$$\mathbf{q}^2 \ll \frac{M_h^2}{\ln \frac{\Lambda^2}{\mu^2}} \tag{3.4}$$

that goes, indeed, into the single point $(q_o, \mathbf{q}) = 0$ in the continuum limit $\Lambda \to \infty$. However, in the cutoff theory this region defines the *non-relativistic* limit $|\mathbf{q}| \ll m$ where m is the mass of the quanta in the condensate (see Eq. (3.3)).

In this regime, any scalar condensate, whatever its origin may be, is a highly correlated structure with long-range order due to the coherence effects associated with the phase of the non-relativistic condensate wave-function [22, 23]. Therefore, for $\mathbf{q} \to 0$, the deviations from a Lorentz-covariant energy spectrum $\tilde{E}(\mathbf{q}) = \sqrt{\mathbf{q}^2 + M_h^2}$ are not surprising.

Now, for $M_h = \mathcal{O}(10^2)$ GeV, it is a matter of taste to decide whether, for instance, values $|\mathbf{q}| \ll 10^{-5}$ eV/c (corresponding to wavelengths much larger than 1 cm) may be considered infinitesimal or not. In the case of a positive answer, these deviations from exact Lorentz-covariance on such scales should be taken seriously. Indeed, as discussed in [18], the associated extremely weak $1/r$ potential does not disappear when coupling the scalar fields to gauge bosons.

References

[1] A complete review of the 'triviality' issue, with extensive reference to the original papers, can be found in: R. Fernández, J. Fröhlich and A. Sokal, *Random Walks, Critical Phenomena and Triviality in Quantum Field Theory* (Springer Verlag 1992).

[2] M. Consoli, in *Gauge Theories Past and Future - in Commemoration of the 60th Birthday of M. Veltman*, R. Akhoury, B. De Wit, P. van Nieuwenhuizen, H. Veltman Eds., World Scientific 1992, p.81; V. Branchina, M. Consoli and N. M. Stivala, Zeit. Phys. **C57**, 251 (1993).

[3] M. Consoli and P.M. Stevenson, Zeit. Phys. **C63**, 427 (1994).

[4] U. Ritschel, Zeit. Phys. **C63**, 345 (1994).

[5] A. Agodi, G. Andronico and M. Consoli, Zeit. Phys. **C66**, 439 (1995).

[6] A complete review of the subject, with extensive reference to the original papers, can be found in : C. B. Lang, "Computer stochastics in scalar quantum field theory", contribution to *Stochastic Analysis and Application in Physics*, Proc. of the NATO ASI, Funchal, Madeira, August 1993, ed. L. Streit (Kluwer Publ. 1994).

[7] J. M. Cornwall, R. Jackiw and E. Tomboulis, Phys. Rev. **D10**, 2428 (1974).

[8] M. Consoli and P. M. Stevenson, Mod. Phys. Lett. **A11**, 2511 (1996).

[9] M. Consoli and P. M. Stevenson, Phys. Lett. **B391**, 144 (1997).

[10] M. Consoli and P. M. Stevenson, Int. J. Mod. Phys. **A15**, 133 (2000).

[11] P. Cea, M. Consoli and L. Cosmai, Mod. Phys. Lett. **A13**, 2361 (1998); P. Cea, M. Consoli, L. Cosmai and P. M. Stevenson, Mod. Phys. Lett. **A14**, 1673 (1999).

[12] P.M. Stevenson and R. Tarrach, Phys. Lett. **B176**, 436 (1986)

[13] U. Ritschel, Phys. Lett. **B318**, 617 (1993).

[14] A. Okopinska, Phys. Lett. **B375**, 213 (1996).

[15] K. Symanzik, Comm. Math. Phys. **16**, 48 (1970).

[16] A. Z. Patashinsky and V. L. Pokrovsky, *Fluctuation Theory of Phase Transitions*, Pergamon Press, London. Italian translation, Editori Riuniti, Roma 1985, pag.146.

[17] R. Anishetty, R. Basu, N. D. H. Dass and H. S. Sharatchandra, Int. J. Mod . Phys. **A14**, 3467 (1999).

[18] M. Consoli and F. Siringo, hep-ph/9910372; M. Consoli, hep-ph/0002098.

[19] F. Ferrer and J. A. Grifols, hep-ph/0001185

[20] This is in agreement with rigorous quantum-field theoretical results on vacuum stability that prescribe the uniqueness of the vacuum in the presence of a *finite* mass gap in the symmetric phase [21]. Just for this reason, m and M_h cannot scale uniformly in the continuum limit if one wants to obtain spontaneous symmetry breaking.

[21] J. Glimm and A. Jaffe, *Quantum Physics: A Functional Integral Point of View*, Springer, New-York, 2nd Ed. 1987.

[22] P. W. Anderson, Rev. Mod. Phys. **38**, 298 (1966); *Basic Notions of Condensed Matter Physics*, Benjamin-Cummings, Menlo Park 1984.

[23] K. Huang, in *Bose-Einstein Condensation*, A. Griffin, D. W. Snoke and S. Stringari eds. Cambridge University Press 1995.